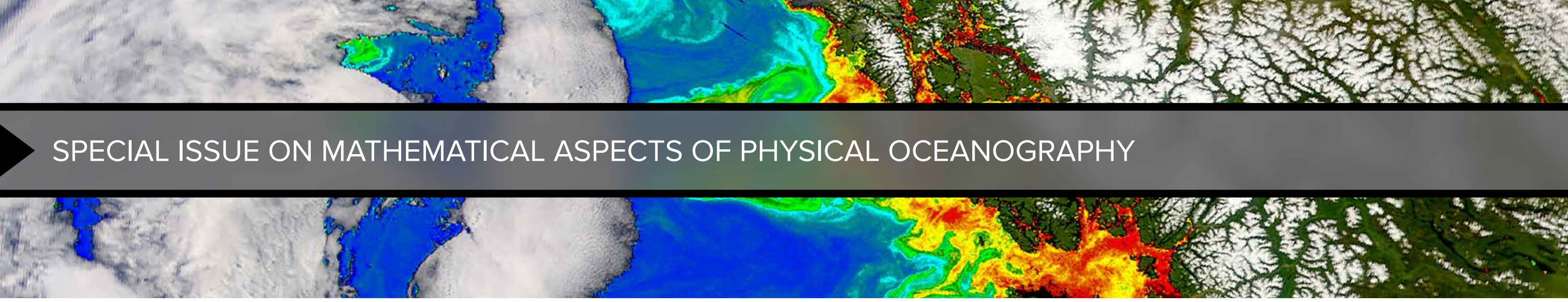

The Value of Asymptotic Theories in Physical Oceanography

By Robin Stanley Johnson

ABSTRACT. Physical oceanography is an exciting, fruitful and important field of study, particularly relevant to the current discourse on, and the effects of, climate change. However, the tried and tested techniques of (and wealth of knowledge available from) classical fluid mechanics seem to have been sidelined, in favor of an emphasis on modeling and numerical methods. In this article, we make the case for returning to fundamental ideas, explaining the essentials of this approach in the context of the Euler (or Navier-Stokes) equation written in a rotating, spherical coordinate system. We support our contention that this is the way forward by presenting (descriptively only) a number of examples that show what can be done, and suggesting that much more is possible. Indeed, we argue that this is the route to be taken before recourse to other, more ad hoc, methods. We will use this approach to provide new insight (and new results) related to the Pacific Equatorial Undercurrent, the Antarctic Circumpolar Current (including the role of exact solutions), and large gyres.

INTRODUCTION

Modern physical oceanography—the study of the physical processes that underpin the motion of the ocean—can be traced back to at least the time of Ekman's seminal work (Ekman, 1905) on his spiral and its role in explaining the observed movement of icebergs. Since that time, many aspects of the flow of our ocean have been examined and interpreted, based, typically, on suitable mathematical models that aim to represent the relevant physical processes; see, for example, Gill (1982), Apel (1987), Tomczak and Godfrey (2003), Segar (2012), and Garrison (2014). Furthermore, starting about the time of the Second World War, a vast amount of quality data have been collected that have provided a significant driver for these investigations. The availability of these data has prompted an upsurge in modeling the various physical processes that are involved in ocean circulation but, perhaps more significantly, also encouraged a much greater role of, and reliance upon, numerical solutions and simulation (Proehl et al., 1986; Lagerloef et al., 1999; Wang et al., 2000; Bell et al., 2016).

Unquestionably, the motion of the ocean in its totality constitutes a very complicated flow system, mainly because so many factors are involved. These include: wind-driven circulation (McCreary, 1985); upwelling and downwelling (G.C. Johnson et al., 2001); effects of Coriolis forces and of associated Ekman transport (Tomczak and Godfrey, 2003; Segar, 2012); atmosphere-ocean interactions (Wu, 1975; Yelland and Taylor, 1996; Davies, 2013; Stewart et al., 2014); temperature, density, and salinity variations (Kessler, 2005); and effects of climate change (Faghmous and Vipin, 2014). In addition, heat balance and transfer in the ocean are important ingredients (Boccaletti et al., 2004; Wallcraft et al., 2008), as are the effects of variable depth and the existence of landmasses that bound the ocean. Superimposed on all this structure are tides and ocean waves (of various types; LeBlond and Mysak, 1978), and the observation that the ocean is predominantly turbulent. All of the above combine to produce a very complex physical system that, it might be thought, can be understood only by numerical and/or physical modeling. That such approaches play a role cannot be denied, but we will argue these should be considered only as a last resort after all the conventional and classical methods have been exhausted; this general philosophy is developed in the essay by Constantin and Johnson (2016a).

Of course, numerical solutions, and numerical simulations, are useful as a means for adding fine detail to flow configurations that are well defined and well understood. Neither numerical solutions nor data analysis can be used as a reasonable basis for developing a fundamental understanding, or reliable interpretation, of such complex flows. Indeed, all the experience gained in fluid mechanics over the last 200 years or so is evidence that we should proceed from simple solutions of the governing equations to more sophisticated developments and extensions. Let us be clear: it would be perverse to ignore the vast body of knowledge (and techniques) relevant to the study of the motion of fluids based on the general governing equations of fluid mechanics. Correspondingly, any recourse to numerical methods or physical modeling should be considered only when all the traditional avenues have been explored and shown to fall short in very significant ways.

The aim of this review is to explain the philosophy behind the approach based on classical fluid mechanics, as it relates to oceanic flows, and to highlight (in outline) some of the successes of these ideas. Further, we will indicate how such methods can be extended and developed, and at what stage (and in what circumstances) numerical and modeling ideas may be useful.

BACKGROUND IDEAS AND METHODS

Any problem involving the motion of a fluid (such as water or air) starts from the general, governing equations of fluid mechanics. These comprise an equation of mass conservation, together with either the Euler equation (for inviscid, but rotational, flows) or the Navier-Stokes equation (for a viscous fluid). Note that we do not have available a reliable theoretical basis for a complete description of fully turbulent flows, but we can incorporate suitable turbulent modeling, if required. In practice, it is usually sufficient to work with the Euler equation for ocean flows, with vorticity included as necessary, because the effects of viscosity are often weak, operating on time and distance scales that are far larger than those associated with the dynamical motions of interest (McCreary, 1985; Maslowe, 1986). Although the dynamic conditions at the free surface should be represented by suitable viscous (wind-stress) action, this can often be modeled by a combination of variable pressure and the transfer of momentum to the inclined face of any surface wave. Furthermore, it is often possible to replace the conditions well below the free surface by decay conditions. Thus, for many discussions of oceanic flows, the Euler equation will suffice, but we must be aware that some special flows do require a significant contribution from the viscous terms in the governing equations; knowledge and experience, and the problem under consideration, will indicate which underlying model for the fluid should be selected. To the chosen system of governing equations we add the appropriate boundary conditions and, for time-dependent problems, a suitable set of initial data; it will be assumed in this context (the ocean) that the resulting problem has a solution and one that is unique. (In some cases, for example, water waves described by the Euler equation, we have theorems that prove that solutions with suitable smoothness properties exist; see, for example, Toland, 1996; Constantin and Strauss, 2004; Constantin and Escher, 2011.) In addition, we should note that a number of explicit solutions, developed within the Lagrangian framework and not involving any approximations, have recently been found; see, for example, Constantin (2012, 2014), Constantin and Monismith (2017), and Henry (2013, 2016). Of course, we would much prefer to obtain an exact solution to our problem, but this is not going to be possible for the type of flows under consideration (except in some very rare and special situations, which we will mention later). So how do we proceed?

The overall plan is to obtain a consistent simplification of the full problem, guided by some general principles. At their simplest, these problems might be based on what we believe is important in the system and what is reasonably accessible in a technical sense. However, this approach is useful only if it can be accomplished in a mathematically consistent way, and this is best done by nondimensionalizing and introducing suitable parameters: the well-tried approach in fluid mechanics. Thus we select, for example, suitable speed and length scales appropriate to the problem to be examined; for unsteady problems, this also provides a time scale. The full system of differential equations, boundary and initial conditions (if relevant), is then written in nondimensional variables

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and the (nondimensional) parameters are identified. The unknowns (the velocity field, the pressure field, and the free surface) are functions of the independent variables and also of the parameters. In most cases, one (or more) of these parameters will be small (or large); indeed, the choice of scales is made—wherever possible—to ensure that this is the result of the nondimensionalization. The procedure is then to construct an asymptotic solution, using the small (or large) parameter as the basis for the asymptotic sequence that defines the asymptotic expansions. Of course, it is often the case that asymptotic solutions are not uniformly valid (i.e., the validity is not across the full time or spatial domain) and so a rescaling of the variables might be necessary. A classic example in fluid mechanics is the boundary-layer scaling near a fixed, solid boundary for large-Reynolds-number flows. (In the rare situation that the parameter we want to use is not small [or large], then the standard procedure is to construct, in the first instance, an asymptotic solution for small/large values of the parameter. This then presents an important role for a numerical approach: starting from the asymptotic solution, the method involves iteration, gradually increasing/decreasing the parameter value.) One final general observation: most problems in oceanography involve more than one parameter, for example, ocean depth/Earth radius, a rotation parameter, and, perhaps, wave amplitude/wavelength. All such parameters are necessarily independent. The mathematically robust and consistent approach involves fixing all the parameters but one, and seeking an asymptotic solution based on just this one parameter. The procedure—the result of a limiting process that underpins the construction of the asymptotic solution—is easily explained; we will describe what this involves and what possible pitfalls might be encountered.

As just mentioned, the simplest procedure is to fix all the parameters but one, labeled, say, ε, and consider the problem of generating a solution valid for $\varepsilon \to 0$. The first important point to make is that the sizes of all the other (independent) parameters are immaterial when we invoke $\varepsilon \to 0$: whether they are, typically, 10^{-5}, 1, or 10^{5}, they remain fixed as the limit process $\varepsilon \to 0$ is imposed. The important outcome is that the sequence of reduced problems, generated by the limiting process, is physically meaningful and mathematically accessible. The resulting solution so obtained is, by definition, asymptotic, and so it may not be convergent in the familiar sense. (This is not a serious drawback on two counts: the asymptotic approach aims to extract the mathematical structure of the problem—and it certainly does this—and if numerical estimates are required, then the standard techniques applicable to the interpretation and use of divergent series can be employed; see, for example, Hardy, 1949, and Dingle, 1973.) The quite exceptional successes of this approach to the solution of complex problems, and especially those in fluid mechanics, are well documented; see, for example, Cole (1968), Van Dyke (1975), Chang and Howes (1984), D.R. Smith (1985), Hinch (1991), Holmes (1995), Kevorkian and Cole (1996), and R.S. Johnson (2004).

At this stage in the development of these ideas, we must present a fundamental principle that underpins any attempt to produce model equations. A system of model equations (for some phenomenon) is acceptable, we submit, only if it can be derived from some underlying equations or guiding principle. In some fields of study, we may have only general guidelines, but in others—and anything associated with the motion of fluids is certainly in this category—we have a set of overarching, governing equations. In this situation, any model equations, if they are to be reliable and to be believable, must be derived by following some precise mathematical procedure. Because we have parameters, and a necessary functional dependence on variables and parameters, the only convincing and robust way forward is to produce an approximate system, carefully constructed and consistent with all the equations and boundary conditions. Any model equations that cannot be so constructed carry no weight, and deserve no consideration, and certainly any conclusions based on such equations must be treated with some skepticism.

APPLICATION TO OCEAN FLOWS

The development of the nondimensional form of the Navier-Stokes (or Euler) equation, written in a rotating, spherical coordinate frame, follows the ideas outlined in the preceding section. This results in a system of governing equations that involves a number of nondimensional parameters, typically: suitable ocean depth/Earth's radius, rotation parameter (measuring the rate of Earth's rotation), and two Reynolds numbers (based on a vertical and a horizontal eddy viscosity). The boundary conditions might add to this a parameter that measures the strength of the wind stress at the free surface, and initial data might add a parameter associated with the motion's wavelength. The existence of a pycnocline/thermocline also introduces a density-ratio parameter. This formulation and associated definitions lead to a number of side issues.

Firstly, no matter what we choose as the ocean depth, the ratio of ocean depth to Earth's radius is always very small, and this ratio is the fundamental parameter in this type of problem; let us label it ε. If we take an average ocean depth as, say, 4 km, then this parameter is about 6×10^{-4}. On the other hand, it might be more appropriate to use the depth scale that produces a vertical-eddy-viscosity Reynolds number of 1 (which is relevant to Ekman flows, for example), and then we have $\varepsilon \approx 10^{-5}$. In this latter case, it follows that the other eddy viscosity (horizontal) gives (Reynolds number)$^{-1} \propto \varepsilon^2$; correspondingly, for Ekman-flow problems, we must have (for consistency) a wind-stress forcing that is independent of ε (recent work of author Johnson and A. Constantin). Another choice for the depth scale might be the average depth of a thermocline. Secondly, for small ε, consistency with the equation of mass conservation (and also with the kinematic condition at the free surface) requires that the vertical component of the fluid velocity is proportional to ε, or possibly smaller. (The case of a smaller velocity component arises in flows that are almost purely rotational, as in large ocean gyres, for example; see Constantin and Johnson, 2017b, and R.S. Johnson, 2017.) It turns out that these various problems—obviously simpler if we work with an inviscid fluid—can be solved to obtain relevant solutions simply by imposing the single asymptotic limiting process: $\varepsilon \to 0$, keeping all the other parameters fixed. This is the shallow-water or thin-shell approximation.

Most importantly, we argue that there is no need to choose an "approximate" coordinate system (such as the tangent plane), nor to invoke any assumptions about the size of the other parameters (including the wavelength of any wave motions; see Constantin and Johnson, 2015, and R.S. Johnson, 2015). Of course, there may be situations (for comparison with earlier work, for example) where a simplified coordinate system (perhaps in conjunction with the f- or β-plane approximation) is appropriate, but otherwise we employ the techniques and ideas precisely as outlined above. Certainly, if an analysis based on the correct representation (rotating, spherical coordinates) of the equations of fluid mechanics is possible, then this should always be the starting point. If this fails to produce reasonable or suitable solutions, then we might revert to some appropriate (ad hoc) modeling or numerical method; we will write more on this later.

Finally, the fundamental approximation ($\varepsilon \to 0$) that we advocate (and the only necessary one, although we might find additional ones useful in order to simplify some of the calculations) is particularly significant. All the coefficients in the governing equations (and boundary conditions) that depend on r' (the radial coordinate; see below) become polynomial approximations for small ε, and these terms remain strictly bounded: the asymptotic expansions are uniformly valid. However, each derivative in r' becomes proportional to ε^{-1} and this has important consequences for the structure of the problem. For example, it shows that the variation in pressure in the radial direction is proportional to ε—a familiar hallmark of the shallow-water approximation—and this property controls the types of solutions that are available and admissible. Although the small-ε asymptotic approximation plays a very important role, we still retain the essential geometry of the problem (i.e., as a spherical shell) with variations in both the azimuthal and meridional directions. An important corollary of this approach is that there is usually no need to impose any further simplifications or restrictions on the underlying geometric structure that describes oceanic flows. With all of these background ideas in place, we now present a few examples, but described in outline only, avoiding any of the technical detail.

EXAMPLES OF OCEANIC FLOWS

We will examine three very different types of flow problems, with the intention of showing how the standard approach in fluid mechanics is readily applicable to different flow configurations; indeed, this is one of the great strengths of the application of such conventional methods. In summary, we will look at a three-dimensional flow of the type observed in the neighborhood of the Equatorial Undercurrent (EUC), two exact (but rather special) solutions that are related to the EUC and to the Antarctic Circumpolar Current (ACC), and a representation of large gyres. In these examples, we present the simplest description by working solely from the Euler equation; we will briefly mention, later, some more recent work based on the Navier-Stokes equation. Because we will need to refer to the coordinates, we start with a defining figure: the spherical coordinate system (Figure 1). Thus, we have the radial coordinate, r' (the prime denoting a physical—dimensional—variable), ϕ the azimuthal angle, and θ the meridional angle (measured from the South Pole, at $\theta = 0$, to the North Pole at $\theta = \pi$).

Example 1

The first example (the three-dimensional EUC) is intended to show how the standard approach used in fluid mechanics can extend a fairly familiar model problem: the EUC described within the β-plane approximation. In this case, we

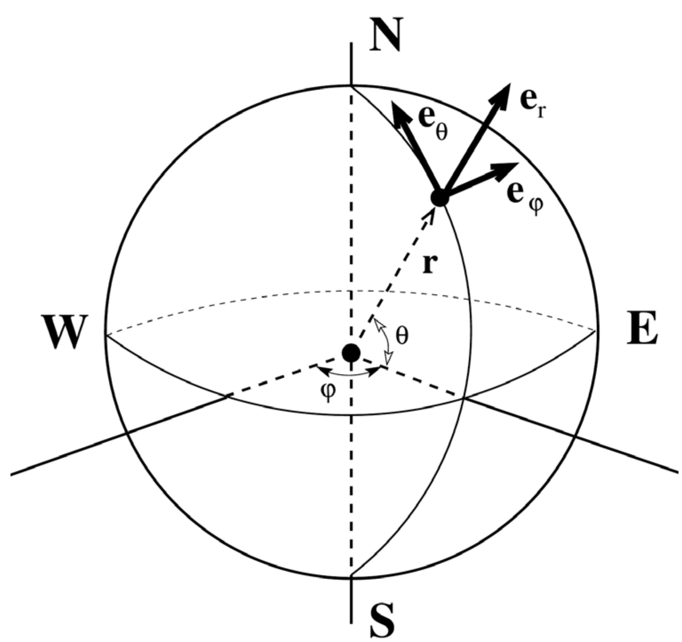

FIGURE 1. The spherical coordinate system. Earth rotates about the north-south axis from west to east.

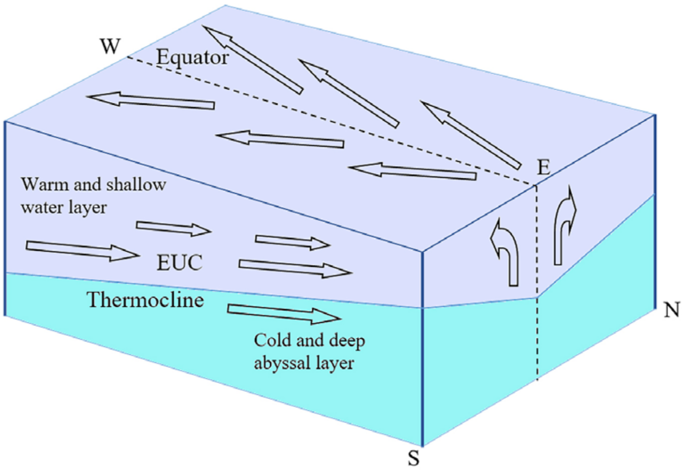

FIGURE 2. Sketch of the flow in the neighborhood of the Pacific equator. EUC = Equatorial Undercurrent.

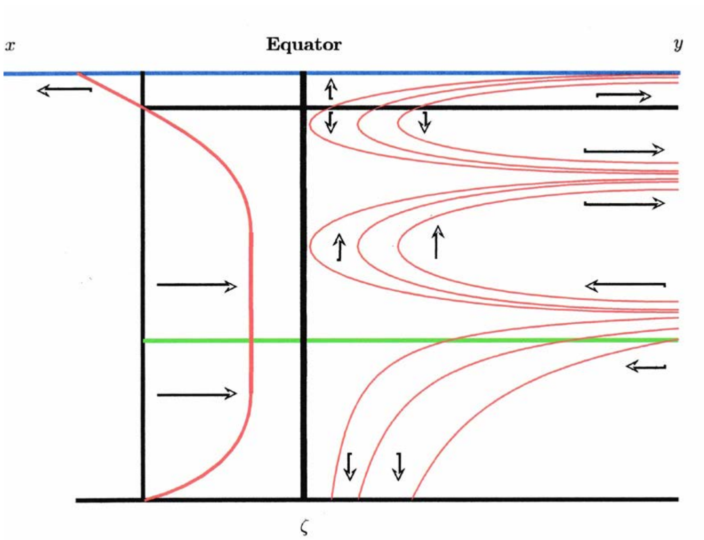

FIGURE 3. Plots depicting the three-dimensional nature of the flow field. The thick vertical line is the equator; to the left of this is the chosen velocity profile; to the right (repeated to the left in the flow field) are the streamlines for a three-cell solution projected on to the y-ζ plane. The arrows denote the direction of the flow; the resulting flow is a combination of these two elements.

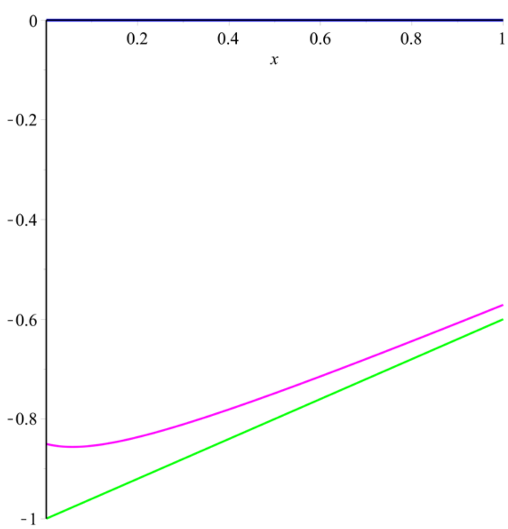

FIGURE 4. The chosen path of the maximum azimuthal speed (green) and the resulting path of the thermocline (magenta). The x-axis is the free surface, plotted from west (x = 0) to east (x = 1).

take as the small parameter the ratio of the depth scale (a typical depth of the thermocline) to the length scale over which variations occur in the azimuthal direction. Thus, we are examining the problem of "slow" variation in the equatorial direction (labeled x here), with a limited region of validity either side of the equator (by virtue of the β-plane approximation). The corresponding leading-order problem (of what turns out to be a uniformly valid asymptotic solution) is still fairly involved, although it is not impossible to analyze it. However, its complications tend to obscure the essential properties of the flow, and so some (reasonable) simplifying assumptions are introduced, including the choice that the free surface and the thermocline both follow the curvature of Earth away from the line of the equator, there is small density change across the thermocline (which is always the case), and there is a (relatively) weak rotation rate for Earth's spin; for more details, see Constantin and Johnson (2017a) and R.S. Johnson (2017). Surprisingly, even with these simplifications, the remaining system of equations allows a lot of choice, producing many different types of three-dimensional flow patterns in the regions close to, and either side of, the line of the equator. In particular, we may choose the velocity profile from the surface downward, and so we can accommodate the westward flow near the surface and a higher-speed flow deeper down, centered more or less at the level of the thermocline, moving eastward (modeling the observed properties of the EUC); see the sketch in Figure 2. In addition, we may choose the path of maximum speed in the undercurrent (to the east), and also adjust the various parameters and free constants to produce any number of cells on either side of the line of the equator. Figure 3 shows an example of the resulting three-dimensional flow, and Figure 4 shows an example of the path of the thermocline, from west to east. (A related problem, based on the f-plane approximation, analyzes the properties of linear waves of arbitrary wavelength on the EUC; see Constantin and Johnson, 2015, and R.S. Johnson, 2017.) This three-dimensional solution cannot, as it stands, accommodate the inclusion of the landmasses at the eastern and western ends of the Pacific equator; this would involve, we suggest, some form of boundary layer near these ends. Furthermore, because this solution has been generated in the β-plane approximation, we are unable to close the cells at any considerable distance away from the line of the equator.

Example 2

As is sometimes the case, the construction of an asymptotic solution indicates the existence of exact solutions; that is what occurred here. We should comment that exact solutions in fluid mechanics, although of quite exceptional usefulness when the basis for further development, relate to very special (and very idealized) flows. We describe two examples of exact solutions of the original, governing equations (inviscid version); in these two cases we can allow any physically realistic velocity profile, varying with depth, and also an appropriate surface pressure distribution that can be adjusted to accommodate any suitable surface profile.

In the first case, which is driven by the usual body force (gravity), we have a vertical velocity component w' that takes the form $w' = F(r' \sin\theta)$ for the arbitrary function F. We can therefore choose F so that the profile corresponds to that observed in the EUC (and so this would be a solution appropriate near the equator); an example is shown in Figure 5. Of course, such an exact solution relates to a flow that moves around the whole globe (with unchanging form), and to embed this solution within the flows observed on Earth we would need to add some transition regions to allow the landmasses at the ends of the Pacific equator. (More details are given in Constantin and Johnson, 2016b, and R.S. Johnson, 2017.)

In the second example of an exact solution, we apply the general approach (as above for the EUC) to the ACC (Figure 6); this case is, however, different in a number of respects. This flow encircles the globe around the polar axis—the only flow of this type that does—but it also comprises a small number of high-speed jets (see, e.g., Ivchenko and Richards, 1996; Rintoul et al., 2001; Olbers et al., 2004; Firing et al., 2011). A jet of restricted (but arbitrary) meridional extent, with a profile that decays with depth, is possible only if the body force is nonconservative (Gallego et al., 2004; K.S. Smith and Marshall, 2009; Stewart et al., 2014); a conservative body force can never produce a physically realistic solution. Figure 7 shows a typical velocity profile, which has been constructed by making a suitable choice of the surface profile and of the (nonconservative) body force; see Constantin and Johnson (2016c), R.S. Johnson (2017).

Example 3

Our final example, and arguably the most exciting and successful to date, is the construction of solutions of the full Euler system that corresponds to large gyres as observed in our ocean. Only one overarching assumption is required: the shallow-water (thin-shell) approximation. However, associated with this assumption, and a choice available to us, is the requirement that the flow (at leading order as $\varepsilon \to 0$) be purely rotational, implying that the vertical velocity component is smaller than $O(\varepsilon)$. In addition, the small ($O(\varepsilon)$) derivative of the pressure in the vertical direction produces (again, at leading order) no variation of pressure in this direction. The resulting reduced (steady) problem is fully nonlinear, and retains all the Coriolis terms; complete analytical solutions of this new system can be constructed (Constantin and Johnson, 2017b; R.S. Johnson, 2017). There is sufficient freedom in the solutions to admit rotational flows (gyres) that sit in either hemisphere and possess bounding streamlines, outside which we may impose zero motion. These flows exist in the spherical shell on the surface of a rotating Earth; we show two examples in Figure 8.

FURTHER AND FUTURE DEVELOPMENTS

The ideas and examples presented here, based on the general, governing equations of fluid mechanics, show what is possible using classical methods. It is, we suggest, quite surprising (and gratifying) that so much can be achieved using this well-tried approach, even for complicated flows of the type observed in our ocean. This evidence would suggest that our original claim carries some weight: to treat these oceanic problems as exercises in

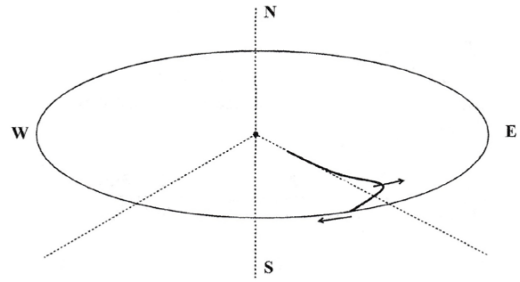

FIGURE 5. A schematic of an EUC-type velocity profile at the equator in an ocean on a rotating Earth.

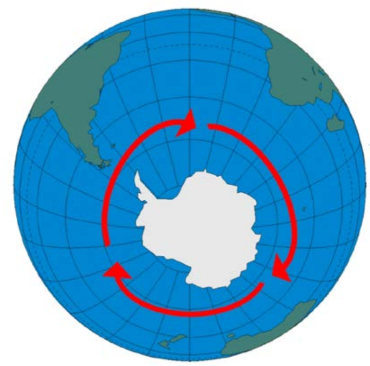

FIGURE 6. Sketch indicating the path of the Antarctic Circumpolar Current.

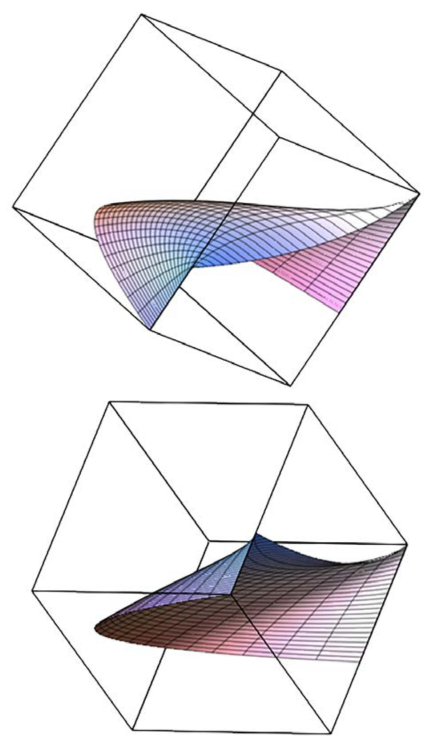

FIGURE 7. Two views of the same exact solution of a jet-like velocity profile (drawn in the Southern Hemisphere) and represented as a surface. The nearest edges are the zeros on the bottom (which follows Earth's curvature) and along the outer edges of the jet; the flow at the surface is chosen as a parabolic-like profile with the maximum speed at the center.

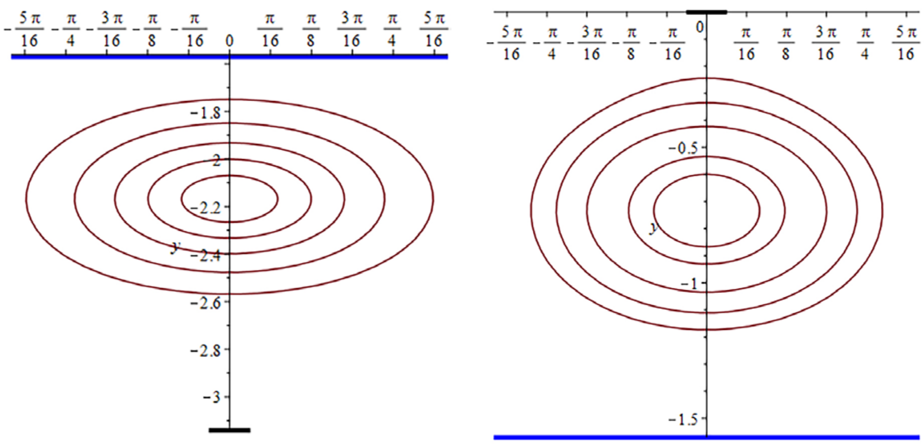

FIGURE 8. Streamlines for constant vorticity flow in the Northern Hemisphere (left panel) and in the Southern Hemisphere (right panel). These gyres are restricted by a bounding streamline, which is the outermost streamline drawn here. The blue line indicates the equator, and the center of the heavy black line is the relevant pole.

classical fluid mechanics before resorting to modeling or the application of numerical methods. In particular, we described how the standard asymptotic approach, used in conjunction with the Euler equation, has produced solutions that relate to the EUC (both a three-dimensional structure and an exact solution for the velocity profile), an exact solution for an ACC jet-like flow, and gyres of any size sitting on a rotating sphere. Further, these solutions appear to be new and certainly add to our knowledge of these flow fields. We also note that these problems have been solved by constructing a reduced system (which is strictly asymptotic) from the original equations; indeed, higher approximations are readily accessible, if they are required. There can be no doubt that other solutions, describing various types of oceanic phenomena, are readily accessible by following the same general route. (Indeed, some current work involves using the Navier-Stokes equation to examine the role of viscosity in large gyres, and also to obtain more general solutions, with a coherent and consistent description, of Ekman-type; three-dimensional flows of EUC-type are being reconsidered without recourse to the β-plane approximation.) The important question, though, is how we can build on these solutions, and so extend their applicability and relevance to oceanic flows.

The inclusion of some additional effects, such as variable depth or density variations, are fairly easily accomplished by invoking multiple-scale methods (i.e., allowing these changes to evolve on some appropriate slow scale). This technique is particularly useful in extending the applicability and relevance of exact solutions. In the case of a more rapid change in depth, for example, we have a natural opening for a numerical approach: start with the multiple-scale (asymptotic) solution and then gradually adjust (iterate) to accommodate more rapid changes. Thermoclines present no difficulties, as they can be included in the type of formulation already mentioned (see, e.g., Constantin and Johnson, 2015; R.S. Johnson, 2017). Wind-driven waves, on the other hand, generally require a broader approach. This is the type of problem where an element of modeling is very useful—possibly essential. The mechanisms for the action of the wind come in many different forms, although for a classical viscous fluid we could simply be given a surface shear stress. This, however, may not satisfactorily describe the transfer of energy to the water: we may wish to represent the complexity of this (turbulent) motion by a suitable model that describes the processes involved. Furthermore, the inclusion of variable eddy viscosities should not prove too difficult, possibly using multiple scales again. All the above—and no doubt others could be added to the list—either use the conventional fluids approach directly (invoking multiple scales as expedient) or incorporate some element of modeling. It is clear that this approach never starts from a set of model equations. At most we have a reduced system that has been derived from, and is consistent with, a set of general, governing equations.

The analytical (asymptotic) ideas, including the method of multiple scales, are very useful and very powerful (and also very reliable and robust), but they struggle if many different additional elements are brought together. When one or two are included, we can expect to obtain detailed analytical results, but if more are added, then a numerical approach (direct integration or simulation), guided by a combination of these results, might give a broader and more comprehensive solution of complex oceanic flows.

In conclusion, we have attempted to make a case for using classical and conventional fluid mechanics to a far greater extent than appears to be the case hitherto. Let us be clear: we do not advocate the rejection of modeling or numerical methods—that would be unthinkable—but they should be relegated to a subsidiary role, following careful (and extensive) analytical investigations. Analytical (asymptotic) methods should rightly be set aside if they do not generate useful results, or because the mathematical technicalities are insurmountable, but the evidence presented here suggests that much can be done with them. Indeed, we have barely scratched the surface of what is possible: there will be many other oceanic flows that can be analyzed and described using classical fluid mechanics. The investigations are ongoing.

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